MATH 501
Discrete Mathematics
Lecture 1:
Basics of propositional calculus

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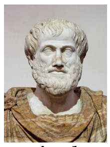
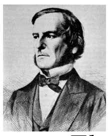
1 Inception of modern logic

1.1 Quick historical overview

From syllogistics to formal logic



- For about 2000 years, the *Syllogism* by Aristoteles (Αριστοτέλησ, 384–322 BC) was the *most exact* form of logics.



- George Boole (1815–1864) develops in "The Mathematical Analysis of Logic" (published 1847) a *formal decision procedure* for truth in propositional logic. He develops *conjunctive* and *disjunctive normal form*.

- The publication of the "Begriffsschrift" by Gottlob Frege (1848–1925) in 1879 introduces *formal language* and *formal proofs* into logics. It revolutionizes logic and is the *basis* of modern computer science.

Intrinsics of formal logic

Note:

- We don't *actually* care for truth
- Logics is a set of *syntactic entities* and *rules* on how to manipulate them.

Example

A journalist asks a man at his 100$^{\text{th}}$ birthday: "What is the secret for having such a long life?".

The answer:
"I keep a strict diet: Whenever I do not drink beer at a meal, I have fish. However if I have fish and beer for the same meal, I abstain from eating ice-cream. In case I eat ice-cream or do not drink beer, I avoid fish."

Can you *simplify* this?

2 Propositional calculus

2.1 Syntax and notation

Basic syntax of propositional logic

- We assume a set of atomic formulas (atoms) or propositions a, b, ...
- We use the usual propositional connectives:

$$\wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad \neg \quad \bot \quad \top$$

Definition 1 (Formula)**.**
- Every atom is a formula
- $\top$ and $\bot$ are formulas
- If F is a formula, then $\neg F$ is a formula
- If F and G are formulas and $\otimes$ is a binary connective, then $(F \otimes G)$ is a formula

Basic syntax of propositional logic

Examples:

Propositional logic formula
$(\neg(a \vee b) \rightarrow \bot)$ is a formula, because:

- a and b are formulas because a and b are atoms.
- Therefore, $(a \vee b)$ is a formula, because $\vee$ is a connective.
- Therefore, $\neg(a \vee b)$ is a formula, because $\neg$ is a unary connective.
- $\bot$ is a formula.
- Therefore $(\neg(a \vee b) \rightarrow \bot)$ is a formula, because $\rightarrow$ is a connective.

2.2 Semantics of propositional logic

Valuation

Definition 2 (Truth values, Valuation)**.** Let D be a subset of the set of atomic formulas. A function

$$\mathcal{A} : D \rightarrow \{t, f\}$$

from the set D to the set $\{t, f\}$ of truth values is called a valuation of D.

Valuation: Example

Example 3. The atoms of the formula $(\neg(a \vee b) \to \bot)$ are a and b. Therefore take $D = \{a, b\}$ as the domain of possible valuations.

Four valuations are possible:

Valuation	a	b
$\mathcal{A}_0$	$\boldsymbol{f}$	$\boldsymbol{f}$
$\mathcal{A}_1$	$\boldsymbol{f}$	$\boldsymbol{t}$
$\mathcal{A}_2$	$\boldsymbol{t}$	$\boldsymbol{f}$
$\mathcal{A}_3$	$\boldsymbol{t}$	$\boldsymbol{t}$

(*e. g.*, valuation $\mathcal{A}_1 = \{(a \mapsto \boldsymbol{f}), (b \mapsto \boldsymbol{t})\}$)

Semantics of propositional logic

Definition 4 (Interpretation)**.** Let $E \supseteq D$ be the set of all formulas that can be constructed from the atoms in D. An interpretation $\mathcal{I} : E \to \{\boldsymbol{t}, \boldsymbol{f}\}$ is an extension of $\mathcal{A}$ such that

- For each atom $a \in D$, $\mathcal{I}(a) = \mathcal{A}(a)$,
- $\mathcal{I}(\top) = \boldsymbol{t}$,
- $\mathcal{I}(\bot) = \boldsymbol{f}$,
- $\mathcal{I}(\neg F) = \begin{cases} \boldsymbol{t}, & \text{if } \mathcal{I}(F) = \boldsymbol{f} \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$,

$\vdots$

Semantics of propositional logic

Definition 5 (Interpretation (continued))**.** $\vdots$

- $\mathcal{I}((F \wedge G)) = \begin{cases} \boldsymbol{t}, & \text{if } \mathcal{I}(F) = \boldsymbol{t} \text{ and } \mathcal{I}(G) = \boldsymbol{t} \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$,
- $\mathcal{I}((F \vee G)) = \begin{cases} \boldsymbol{f}, & \text{if } \mathcal{I}(F) = \boldsymbol{f} \text{ and } \mathcal{I}(G) = \boldsymbol{f} \\ \boldsymbol{t} & \text{otherwise.} \end{cases}$,
- $\mathcal{I}((F \to G)) = \begin{cases} \boldsymbol{t}, & \text{if } \mathcal{I}(F) = \boldsymbol{f} \text{ or } \mathcal{I}(G) = \boldsymbol{t} \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$,
- $\mathcal{I}((F \leftrightarrow G)) = \begin{cases} \boldsymbol{t}, & \text{if } \mathcal{I}(F) = \mathcal{I}(G) \\ \boldsymbol{f} & \text{otherwise.} \end{cases}$.

Interpretation example

Example:

Interpreting a propositional logic formula
Let $F = (\neg(a \vee b) \to \bot)$ be a formula and $\mathcal{A} = \{(a \mapsto \boldsymbol{f}), (b \mapsto \boldsymbol{t})\}$ be a valuation for F.

- a and b are atoms, so $\mathcal{I}(a) = \mathcal{A}(a) = \boldsymbol{f}$ and $\mathcal{I}(b) = \mathcal{A}(b) = \boldsymbol{t}$,
- Therefore, $\mathcal{I}((a \vee b)) = \boldsymbol{t}$,
- Therefore, $\mathcal{I}(\neg(a \vee b)) = \boldsymbol{f}$,
- $\mathcal{I}(\bot) = \boldsymbol{f}$ according to definition,
- Therefore $\mathcal{I}((\neg(a \vee b) \to \bot)) = \boldsymbol{t}$.

Synopsis

Notes:

- We will use the terms valuation and interpretation *interchangeably*. The distinction only serves for a *rigorously correct* definition.
- This is a formal definition. Intuitively (but with *less rigor*), one could use truth tables to describe the effect of the connectives.
- So far we have syntax and interpretation. We do *not* yet have *rules* for transformation (*e. g.*, Boolean algebra).

Boolean algebra

A Boolean Algebra must satisfy these axioms:

$x \vee \bot = x$	$x \wedge \top = x$	
$x \vee \top = \top$	$x \wedge \bot = \bot$	
$x \vee x = x$	$x \wedge x = x$	
$x \vee \neg x = \top$	$x \wedge \neg x = \bot$	
$\neg\neg x = x$		
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	Commutativity
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	Associativity
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	Distributivity
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	DeMorgan's Law

Validity and satisfiability

Definition 6 (Model, satisfiability, validity)**.** Let F be a formula and $\mathcal{A}$ be a valuation that is defined for all atoms $a_0, \ldots, a_n$ in F.

- $\mathcal{A}$ is called a model of F iff $\mathcal{A}(F) = \boldsymbol{t}$.
- A formula F is called satisfiable iff there is a model $\mathcal{A}$ for F.
- A formula F is called valid iff every valuation $\mathcal{A}$ for F models F.
- We write $\mathcal{A} \models F$ if $\mathcal{A}$ *models* F, otherwise we write $\mathcal{A} \not\models F$.
- We write $\models F$ if F is *valid*, or $\not\models F$ if F is *not satisfiable*.

Validity and satisfiability

Definition 7 (Equivalence)**.** Let F and G be formulas. F and G are equivalent, iff for all valuations $\mathcal{A}$ we have that $\mathcal{A}(F) = \mathcal{A}(G)$.

- For two equivalent formulas F and G we write $F \equiv G$.
- Do *not* mix up equivalence with equality!
- Note that even formulas with *different sets of atoms* can be equivalent (*e. g.*, valid formulas).

Semantics of propositional logic

Notes:

- We can check satisfiability/validity/equivalence though *truth tables*.
- Truth tables offer an algorithmic way, which can be performed *automatically*
- But there is a *downside*: For a formula with only *100 atoms*, we need a truth table with 1267650600228229401496703205376 rows (1.27×10^{30}).
- In defense of truth tables: There *is no better way* (NP completeness of SAT).

Example

Recall the dietary instructions for longevity

The secret of longevity

"I keep a strict diet: *Whenever* I do *not* drink *beer* at a meal, I have *fish*. However *if* I have *fish and beer* for the same meal, I *abstain from* eating *ice-cream*. *In case* I eat *ice-cream or* do *not* drink *beer*, I *avoid fish*."

simplify (just use Boolean algebra):

$$\begin{aligned} & (\neg b \to f) \land ((f \land b) \to \neg i) \land ((i \lor \neg b) \to \neg f) \\ & \vdots \\ \equiv\ & (b \land \neg f) \end{aligned}$$