Appendix for Information Revision: The Joint Revision of Belief and Trust

Ammar Yasser[1] , **Haythem O. Ismail**[2,1]

[1]German University in Cairo, Egypt
[2]Cairo University, Egypt
ammar.abbas@guc.edu.eg, haythem.ismail@guc.edu.eg

Observation 5.1.1 Positive Entrenchment.
If $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$, then, $\mathcal{K} \ltimes (\phi, \sigma) \not\prec_\phi \mathcal{K}$.

Proof. This easily follows from the Supported Entrenchment postulate ($\ltimes_5$). The postulate states that the only way for ϕ, on revision with ϕ, to become less entrenched is for the belief base to be inconsistent before revision. By contraposition, ϕ can not become less entrenched if the belief base is not inconsistent. Hence, if the belief base before revision is consistent $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$, ϕ can not be less entrenched $\mathcal{K} \ltimes (\phi, \sigma) \not\prec_\phi \mathcal{K}$. □

Observation 5.1.2 Positive Persistence.
If $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$ and $\phi \in Cn(For(\mathcal{B}(\mathcal{K})))$, then $\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))))$.

Proof. From Observation 5.1.1, starting with a consistent belief base, on revising with ϕ, it can not become less entrenched. Thus, from the definition of entrenchment:

1. If $\phi \in Cn(For(\mathcal{B}(\mathcal{K})))$, it can not be the case that $\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))))$, and
2. If $(\phi, b_1) \in \mathcal{B}(\mathcal{K})$ and $(\phi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))$, it can not be the case that $b_1{}_b b_2$. Which means that either b_1 is the same as b_2 or $b_1 \prec_b b_2$.

In both cases, it is obvious that $\phi \in For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma)))$ hence, trivially, $\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))))$. □

Observation 5.1.3 Negative Persistence.
If $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K})))$, then $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))))$.

Proof. From the Opposed Entrenchment postulate ($\ltimes_6$), on revising with ϕ, $\neg\phi$ can not become more entrenched. Thus, from the first clause in definition of entrenchment (Definition 3.4) it easily follows that if $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K})))$, it can not be the case that $\neg\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))))$. □

Observation 5.1.4 Formula Relevance.
If $\mathcal{K} \not\equiv_\psi \mathcal{K} \ltimes (\phi, \sigma)$, then ψ is ϕ- or $\neg\phi$-relevant.

Proof. Let $(\psi, b_1) \in \mathcal{B}(\mathcal{K})$ and $(\psi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma))$, then this easily follows given the two cases:

i If $b_1 \prec_b b_2$, then $\mathcal{K} \prec_\psi \mathcal{K} \ltimes (\phi, \sigma)$. Hence, from $\ltimes_9$, ψ is ϕ-relevant.

ii If $b_1 \succ_b b_2$, then $\mathcal{K} \ltimes (\phi, \sigma) \prec_\psi \mathcal{K}$ and hence, from $\ltimes_{10}$, ψ is either ϕ-relevant or $\neg\phi$-relevant.

□

Observation 5.1.5 Trust Relevance.
If $\mathcal{K} \not\equiv_{\sigma', T} \mathcal{K} \ltimes (\phi, \sigma)$, then σ' is ϕ- or $\neg\phi$-relevant.

Proof. Let $(\sigma', T, t_1) \in \mathcal{T}(\mathcal{K})$ and $(\sigma', T, t_2) \in \mathcal{T}(\mathcal{K} \ltimes (\phi, \sigma))$, then this easily follows given the two cases:

1. $t_1 \prec_t t_2$, in which case, σ' is more trusted. From $\ltimes_7$, if a source σ' is more trusted, then, either:
   (a) $\sigma' \neq \sigma$ is supported by ϕ, in which case, σ' is ϕ-relevant; or
   (b) $\sigma' = \sigma$. Since σ conveyed ϕ, it trivially follows that σ supports ϕ and hence it follows that σ' is ϕ-relevant.
2. $t_1 \succ_t t_2$, in which case, σ' is less trusted. From $\ltimes_8$, if a source σ' is less trusted, then, either:
   (a) σ' is $\neg\phi$-relevant where $\sigma' \neq \sigma$, or
   (b) $\sigma' = \sigma$, in which case, it trivially follows that σ' is ϕ-relevant.

□

Observation 5.1.6 No Trust Increase I.
If $\phi \notin For(\mathcal{B}(\mathcal{K} \ltimes (\phi, \sigma)))$, then, there is no source $\sigma' \in \mathcal{S}_\mathcal{K}$ such that $\mathcal{K} \prec_{\sigma, T} \mathcal{K} \ltimes (\phi, \sigma)$.

Proof. This trivially follows from the definition of Positive Relevance ($\ltimes_7$). □

Observation 5.1.7 Rational Revision.
If $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$, then an operator that observes $\ltimes_5$ and $\ltimes_6$ allows for only cases in Table 1 to occur.

Proof. This could be done on a case-by-case basis. The goal is to show that given Table 1, Table 2, Observation 5.1.1, and the Opposed Entrenchment postulate ($\ltimes_6$) for every case:

1. ϕ does not become less entrenched, and
2. $\neg\phi$ does not become more entrenched.

- Case B_1. ϕ was not believed and after revision became believed. Hence, ϕ became more entrenched. On the other hand $\neg\phi$ was not believed and, after revision, it remained not believed. Hence, it did not become more entrenched.
- Case B_2. Neither ϕ nor $\neg\phi$ became more or less entrenched as they were not believed and they remained not believed.
- Case B_3. ϕ became more entrenched and $\neg\phi$ did not become more entrenched.
- Case B_4. Neither ϕ nor $\neg\phi$ became more or less entrenched. ϕ was still believed to the degree it was before and $\neg\phi$ remained not believed as it was before as well.
- Case B_5. In this case, $\neg\phi$ became less entrenched as belief in it ceased to persist after revision compared to before revision. On the other hand, ϕ became more entrenched.
- Cases B_6 and B_7. $\neg\phi$ became less entrenched while ϕ remained the not believed as it was.
- Case B_8. $\neg\phi$ did not become more entrenched. ϕ remained not believed as it was the case before revision.

As for cases in Table 2, they are not allowed because they fail to observe the postulates. In cases B_9, B_{12}, and B_{13}. $\neg\phi$ becomes more entrenched which is not allowed given $\ltimes_6$, while in cases B_{10} and B_{11} ϕ becomes less entrenched which is not allowed Given Observation 5.1.1. □

Observation 5.2.1 Single Source Revision.
If, given information structure $\mathcal{I}$, $\mathcal{S} = \{\sigma\}$, then, for any information state $\mathcal{K}_i$ where $i > 0$, the maximum degree in $\{ t \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}) \}$ is δ.

Proof. **Basis**: Starting with an empty information state $\mathcal{K}_0 = (\{\}, \{\}, ())$, from $\ltimes_2$ we get that, $(\sigma, T, \delta) \in \mathcal{T}(\mathcal{K}_0 \ltimes (\phi, \sigma))$. So in the base case, where the size of $\mathcal{T}(\mathcal{K})$ is 1, δ is the maximum degree of trust for any source.

Hypothesis: Suppose that the maximum degree of $\{ t \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}') \}$ is δ for some $\mathcal{K}'$.

Step: Assume that some information source σ' is more trusted after revision. That is, $\mathcal{K} \prec_{\sigma',T} \mathcal{K} \ltimes (\phi, \sigma)$. By the hypothesis, the maximum degree of trust in a source in $\mathcal{T}(\mathcal{K}')$ is δ. From $\ltimes_9$ there are two possible ways for σ' to be more trusted after revising with ϕ:

1. $\sigma' \neq \sigma$ is supported by ϕ; or
2. $\sigma' = \sigma$ but, there is a σ-independent ϕ-kernel.

The first case requires that there is some $\sigma' \neq \sigma$. In the second case, for a σ-independent ϕ-kernel (Γ) to exist, every $\psi \in \Gamma$ must not be exclusively supported by σ. As there is no source $\sigma' \neq \sigma$, then any $\psi \in \Gamma$ could only be source-supported by σ. Moreover, since the conveyance inclusion filter is non-forgetful, and $\mathcal{B}(\mathcal{K}_0)$ is empty, there is no formula that is supported by a formula not supported by σ. Thus, in all cases, for a source to be more trusted after revision, it is required that there is at least one $\sigma' \neq \sigma \in \mathcal{S}$ and this is not possible as $\mathcal{S}$ contains only σ. Hence, the maximum degree of trust in σ on any topic in $\mathcal{T}(\mathcal{K}' \ltimes (\phi, \sigma))$ will not be greater than δ.

Thus, for any information state $\mathcal{K}$ where $\mathcal{S} = \{\sigma\}$, the maximum degree of trust in σ on any topic will not exceed δ.

□

Observation 5.2.2 No Trust Increase II.
If for every $\sigma \in \mathcal{S}_{\mathcal{K}_i}$, there is no source σ' that is σ-relevant, then, there is no $\sigma \in \mathcal{S}_{\mathcal{K}_i}$ such that $\mathcal{K}_{i-1} \prec_{\sigma,T} \mathcal{K}_i$.

Proof. Let $\mathcal{K}_i = \mathcal{K}_{i-1} \ltimes (\phi, \theta)$. For a source $\sigma' \neq \theta$ to be more trusted after revision ($\mathcal{K}_{i-1} \prec_{\sigma',T} \mathcal{K}_i$), it must be that σ' is supported by the newly conveyed ϕ. However, if formula ϕ supports σ', it must be that the source of ϕ, in this particular revision instance θ, also supports σ'. But, since there is no source that is relevant to σ', by contraposition, ϕ is not relevant to σ'. Hence, on revision with ϕ, no source $\sigma' \neq \theta$ will be more trusted. Moreover, for θ to be more trusted, it must be that there is a θ-independent ϕ-kernel. Similar to the previous proof, for this kernel to exist, every $\psi \in \Gamma$ must not be exclusively supported by θ. Since the conveyance inclusion filter is non-forgetful, and $\mathcal{B}(\mathcal{K}_0)$ is empty, there are no source-less formulas. Subsequently, any $\psi \in \Gamma$ must be supported by a source different than θ. Since there is no source supporting any source, such kernel does not exist and hence $\mathcal{K}_{i-1} \not\prec_{\theta,T} \mathcal{K}_i$. Thus, given that neither the conveyor of ϕ nor any other source different from the conveyor will be more trusted, there is no source σ such that $\mathcal{K}_{i-1} \prec_{\sigma,T} \mathcal{K}_i$. □

Observation 5.2.3 No Trust Increase III.
If for every information state $\mathcal{K}_j$ where $0 < j \leq i$, for every source $\sigma_j \in \mathcal{S}_{\mathcal{K}_j}$, there is no source σ'_j that is σ_j-relevant, then, the maximum degree in $\{ t \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}_i) \}$ is δ.

Proof. **Basis**: Starting with an empty information state $\mathcal{K}_0 = (\{\}, \{\}, ())$, from $\ltimes_2$ we get that, $(\sigma, T, \delta) \in \mathcal{T}(\mathcal{K}_0 \ltimes (\phi, \sigma))$. So in the base case, where $j = 1$, δ is the maximum degree of trust for any source.

Hypothesis: Suppose that the maximum degree in $\{ t \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}_j) \}$ is δ for some $\mathcal{K}_j$ where $0 < j < i$.

Step: Similar to the previous two proofs, assume that some information source σ' is more trusted after revision. That is, $\mathcal{K}_j \prec_{\sigma',T} \mathcal{K}_{j+1}$ where $\mathcal{K}_{j+1} = \mathcal{K}_j \ltimes (\phi, \sigma)$. From $\ltimes_9$ there are two possible ways for σ' to be more trusted after revising with ϕ:

1. $\sigma' \neq \sigma$ is supported by ϕ; or
2. $\sigma' = \sigma$ but, there is a σ-independent ϕ-kernel.

The first case addresses how a source σ', different from the conveyor, could be more trusted. For this to occur, it must be the case that σ' is supported by the newly conveyed ϕ. However, if formula ϕ supports σ', then, the source of ϕ, in this particular revision instance σ, also supports σ'. But, since there is no source that is relevant to σ' in $\mathcal{K}_{j+1}$, by contraposition, ϕ is not relevant to σ' and hence σ' can not be more trusted. Thus, for every $\sigma' \neq \sigma$ and $\sigma' \in \mathcal{S}_{\mathcal{K}}$, $\mathcal{K}_j \not\prec_{\sigma',T} \mathcal{K}_{j+1}$. On the other hand, for the conveyor of ϕ (σ) to be more trusted, at least one σ-independent ϕ-kernel (Γ) must exist. Since the conveyance inclusion filter is non-forgetful, and $\mathcal{B}(\mathcal{K}_0)$ is empty, there is no formula that is not supported by some source. Thus, for every $\psi \in \Gamma$, it must be at least supported by a source $\sigma' \neq \sigma$. Since there is no source supporting any source in $\mathcal{K}_j$, such kernel does not exist and hence $\mathcal{K}_j \not\prec_{\sigma,T} \mathcal{K}_{j+1}$. Thus, the maximum degree of $\{ t_{j+1} \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}_{j+1}) \}$ (t'_{j+1}) is not more than the maximum degree of $\{ t_j \mid (\sigma, T, t) \in \mathcal{T}(\mathcal{K}_j) \}$ (t'_j). Since, given the hypothesis, $t'_j \preceq_t \delta$, then, $t'_{j+1} \preceq_t \delta$.

Hence, the maximum degree of trust in any source in any $\mathcal{T}(\mathcal{K}_j \ltimes (\phi, \sigma))$ where $0 < j \leq i$ is δ.

□

Observation 6.1 $\ltimes_{\mathfrak{G}}$ **observes** $\ltimes_{3-10}$ **of Section 5.2**.

Proof. To construct the proof, we show that $\ltimes_{\mathfrak{G}}$ observes every postulate as follows.

Consistency The consistency postulates insists that the beliefs of a revised information state are consistent. It follows from the definition of $\ltimes_{\mathfrak{G}}$ that whenever a contradiction appears, recursive kernel contraction will occur till the beliefs are consistent.

Resilience The first step in the operation of $\ltimes_{\mathfrak{G}}$ is that if ϕ is inconsistent, it will be rejected. The first condition for any source to be more trusted, is that ϕ, the conveyed formula, must succeed. Since ϕ does not succeed, no source will be more trusted. Hence it trivially follows that σ (the conveyor) can not become more trusted.

Supported Entrenchment Supported Entrenchment, and subsequently, Observation 5.1.1 enforce that ϕ, on revision with ϕ, can not become less entrenched if the belief base before revision is consistent. More formally, it can not be the case that:

I $\phi \in Cn(For(\mathcal{B}(\mathcal{K})))$ and $\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$; or

II $(\phi, b_1) \in \mathcal{B}(\mathcal{K})$, $(\phi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))$, and $b_1 \succ_b b_2$. Given that $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$

For Case I, if $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$, we need to show that if $\phi \in For(\mathcal{B}(\mathcal{K}))$ it can not be the case that $\phi \notin For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma)))$. Since $Cn(For(\mathcal{B}(\mathcal{K}))) \neq \mathcal{L}$, then, $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K})))$. $\ltimes_{\mathfrak{G}}$ adds ϕ, after deriving a degree of belief, to the belief base and performs recursive kernel contraction to eliminate any inconsistencies. Starting from a consistent belief base, it trivially follows that, there will not arise an inconsistency regarding ϕ and hence it will succeed.

Case II can not occur, as well, for the following reason. If $(\phi, b_1) \in \mathcal{B}(\mathcal{K})$ and $(\phi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))$, with $b_1 = Max(F, S_1)$ where F is belief in ϕ due to formulas supporting ϕ and S_1 is the highest degree of trust in a source that conveyed ϕ. Since no new formulas were added, on revising with ϕ that is already believed, $b_2 = Max(F, S_2)$. Let the degree of trust in σ on a topic containing ϕ be t_σ. Then, S_2 is either the same as S_1 if $S_1 \succ_t t_\sigma$ or the same as t_σ otherwise. In both cases, the maximum degree of trust in a source supporting ϕ did not decrease and hence belief will not decrease.

Opposed Entrenchment The goal of Opposed Entrenchment is to guarantee that $\neg\phi$, on revision with ϕ, does not become more entrenched. For $\neg\phi$ not to become more entrenched, it can not be the case that:

I $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K})))$ and $\neg\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$; or

II $(\neg\phi, b_1) \in \mathcal{B}(\mathcal{K})$, $(\phi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))$, and $b_1 \prec_b b_2$.

Similar to the proof of Supported Entrenchment, case I can not occur because $\ltimes_{\mathfrak{G}}$, on revising with ϕ, accepts ϕ then resolves contradictions if they occur. Hence, if $\neg\phi$ was not already in the consequence, ϕ will be accepted, and $\neg\phi$ will remain absent from the consequence.

In the second case, if $\neg\phi$ is already believed, $\ltimes_{\mathfrak{G}}$ will compare ϕ to $\neg\phi$ to decide on which one to remove. If $\neg\phi$ had a lower degree, it will be removed and hence it will not be more entrenched. On the other hand, if ϕ had a lower degree, it will be rejected. However, $\neg\phi$ will not be more entrenched because there was no trust increase, or belief increase, as ϕ did not succeed.

Positive Relevance The postulate states that if a source σ' is more trusted, then ϕ succeeds and either:

I $\sigma' \neq \sigma$ is supported by ϕ; or

II $\sigma' = \sigma$ and there is a $\Gamma \subseteq For(\mathcal{B}(\mathcal{K}))$ that is a σ-independent ϕ-kernel.

For case I, we need to prove that if ϕ succeeds and source $\sigma' \neq \sigma$ becomes more trusted, on revision with $\ltimes_{\mathfrak{G}}$, then, σ' is supported by ϕ. A source σ' is more trusted, according to $\ltimes_{\mathfrak{G}}$, if the support degree of σ' increases after revision. The support degree of a source σ' in an information state $\mathcal{K}$ is the sum of support degrees of formulas in $\sigma'(\mathcal{H}(\mathcal{K}))$ with respect to σ'. Thus, in case I, for σ''s degree of support to increase, the number of σ'-independent kernels for formulas in $\sigma'(\mathcal{H}(\mathcal{K}))$ must have increased. Since, possibly, the only new added formula to the belief base is ϕ, if a new σ'-independent kernel for some $\psi \in \sigma'(\mathcal{H}(\mathcal{K}))$ (Γ_ψ) was introduced, then it must be that $\phi \in \Gamma_\psi$. From Observation 4.1, it easily follows that if ϕ belongs to some ψ-kernel where ψ is a formula conveyed by σ', then ϕ supports σ'.

As for case II, we need to prove that if source σ, the conveyor of ϕ, is more trusted, then the reason was that there is a σ-independent ϕ-kernel that is already believed. For any formula $\psi \neq \phi$ and $\psi \in \sigma(\mathcal{H}(\mathcal{K}))$, it is either the case that 1) $\phi \in \Gamma$ where Γ is a ψ-kernel or 2) ϕ does not belong to any ψ-kernel. In both cases, the support degree of ψ with respect to σ will not increase. Because, in case 1, if ϕ belongs to some ψ-kernel it can not contribute any additional value to its support degree with σ being the new support (σ-dependent support does not contribute to the support degree with respect to σ). Also, in case 2, it trivially follows that the support degree of ψ will not increase. Then, it must be the case that if the support degree of σ increased, the support degree of ϕ with respect to σ was the reason. Finally, for ϕ's support degree with respect to σ to increase, it must be the case that there is a σ-independent ϕ-kernel.

Negative Relevance The postulate represents the two cases that must have happened if a source σ' became less trusted.

I $\phi \in For(\mathcal{B}(\mathcal{K}_{\ltimes}(\phi, \sigma)))$ and σ' is ϕ-relevant; or

II $\sigma' = \sigma$, but, there is $\Gamma \subseteq For(\mathcal{B}(\mathcal{K}_{\ltimes}(\phi, \sigma)))$ where Γ is a $\neg\phi$-kernel.

Since $\ltimes_{\mathfrak{G}}$, observers Consistency, then, if $\phi \in For(\mathcal{B}(\mathcal{K}(\phi, \sigma)))$ and $\neg\phi \in Cn(For(\mathcal{B}(\mathcal{K})))$, then, $\neg\phi \notin For(\mathcal{B}(\mathcal{K}(\phi, \sigma)))$. For any source σ' to become less trusted, the support degrees of formulas conveyed by σ' must decrease. For the support degree of a formula ψ to decrease, given $\ltimes_{\mathfrak{G}}$, it must be that either the number of σ'-independent ψ-kernels decreased or that ψ itself was removed from the belief base. Thus, if the support degree of a source σ' decreased it must be because σ'-relevant formulas were removed. And given that $\ltimes_{\mathfrak{G}}$ removes formulas by performing recursive kernel contraction starting with the reason of inconsistency, in this case $\neg\phi$, if a source suffers a decrease in its support degree it must be $\neg\phi$-relevant.

Belief Confirmation For $\ltimes_{\mathfrak{G}}$ to observe Belief Confirmation, then if $(\psi, b_1) \in \mathcal{B}(\mathcal{K})$ and $(\psi, b_2) \in \mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))$ where $b_1 \succ_b b_2$, it must be the case that ϕ supports ψ.

Where $\psi \neq \phi$, given how $\ltimes_{\mathfrak{G}}$ derives the degree of belief for a formula, let $b_1 = Max(F, S)$ where F is belief due to formulas supporting ψ and S is the highest degree of trust in a source that conveyed ψ in $\mathcal{K}$. For b_1 to increase in $\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma)$, it must be that F increased or that S increased or both increased. If F increased, then the number of ψ-kernels increased after revising with ϕ. If the number of ψ-kernels increased after revising with ϕ, then, ϕ belongs to some ψ-kernel and hence ϕ supports ψ. If S increased, then the maximum degree of trust in a source that conveyed ψ increased. For a source to be more trusted after revision from ($\ltimes_7$), it must be the case that it is supported by ϕ.

Belief Refutation For $\ltimes_{\mathfrak{G}}$ to observe Belief Refutation, then, if $\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma) \prec_{\psi} \mathcal{K}$, it must be that:

1. $\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$ and ψ is $\neg\phi$-relevant;
2. Either $\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$ or $\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma) \prec_{\phi} \mathcal{K}$ and ψ is ϕ-relevant; or
3. $\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma) \prec_{\neg\phi} \mathcal{K}$ and ψ is $\neg\phi$-relevant.

For Case 1, if $\phi \in Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$, then, $\neg\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$ given that $\ltimes_{\mathfrak{G}}$ observers Consistency. For a formula to be less entrenched, given $\ltimes_{\mathfrak{G}}$, either its degree of belief decreased or it was removed from the beliefs. In order for that to occur, it must be that the number of ψ-kernels decreased, or trust in sources that conveyed ψ decreased or both. For trust in a source σ' to decrease given $\ltimes_{\mathfrak{G}}$, some σ'-relevant formulas must have been removed. Thus, in both cases, if formula ψ is less entrenched, some ψ-relevant formulas became less entrenched. Given how $\ltimes_{\mathfrak{G}}$ operates, only $\neg\phi$-relevant formulas could be removed and hence given Observation 4.2, ψ is $\neg\phi$-relevant.

In Case 2, since $\ltimes_{\mathfrak{G}}$ adds ϕ to the belief base and only removes it if there is a contradiction with ϕ having the lower degree, if $\phi \notin Cn(For(\mathcal{B}(\mathcal{K} \ltimes_{\mathfrak{G}} (\phi, \sigma))))$, then recursive kernel contraction was performed starting with ϕ- and ϕ-relevant formulas. Hence, similar to the previous case, since $\ltimes_{\mathfrak{G}}$ decided on removing ϕ-relevant formulas, then a formula that will become less entrenched (ψ), given Definition 4.2, it trivially follows that ψ is ϕ-relevant.

Case 3, is similar to Case 1 in the following way. If ψ becomes less entrenched, then it must be the case that some ψ-relevant formulas were removed. Since the $\ltimes_{\mathfrak{G}}$ operator starts kernel contraction by considering ϕ and $\neg\phi$, if $\neg\phi$ becomes less entrenched, it must be the case that some $\neg\phi$-relevant formulas were removed. Thus, the reason ψ-relevant formulas were removed is that they are also $\neg\phi$-relevant and hence given Observation 4.2, ψ is $\neg\phi$-relevant.

□