Reasoning with Artificial Mental States: An Algebraic Approach

Appendix

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In this document, we provide the proofs for the theorems in the paper *Reasoning with Artificial Mental States: An Algebraic Approach* that has been accepted for presentation at the 6\(^{th}\) Workshop on Formal and Cognitive Reasoning (FCR-2020).

**Proof of Theorem 1**

Let \(\preceq\) be a \(k\) partial order on \(\mathfrak{A}\) which is classical in \(i\) for some \(i \in S\) and \(\mathfrak{S}_{\preceq}(\langle Q_1, ..., Q_k \rangle, S) = \langle Q'_1, ..., Q'_k \rangle\). Then, by Definition 3, there does not exist an order \((a_1, ..., a_k) \leq (b_1, ..., b_k)\) such that \(a_i \neq b_i\) for all \(i\) such that \(1 \leq i \leq k\). Hence, \(F(Q_i) \subseteq Q'_i\). By, Definition 3, there does not exist an order such that \((\{T\}^{i-1} \times \{a\} \times \langle T \rangle^{k-i}) \times (P^{i-1} \times \{b\} \times P^{k-i}) \subseteq \preceq_k\) and \(a_i \neq b_i\). Hence, \(Q'_i \subseteq F(Q_i)\). Therefore, \(Q'_i \subseteq F(Q_i)\).

**Proof of Theorem 2**

In the following, for every \(\mathcal{L}\) valuation \(\mathcal{V}\), let \(\preceq_{\mathcal{TV}}\) be a \(T^{\mathcal{V}}\)-induced order and \(\preceq_{\mathcal{TV}}\) be a \(T^{\mathcal{V}}\)-induced order. Further, let the multiset \(\mathfrak{S}_{\preceq_{\mathcal{TV}}}((\langle A_1 \rangle^{\mathcal{V}}, ..., \langle A_k \rangle^{\mathcal{V}}), S) = (F_1, ..., F_k)\) and \(\mathfrak{S}_{\preceq_{\mathcal{TV}}}((\langle A'_1 \rangle^{\mathcal{V}}, ..., \langle A'_k \rangle^{\mathcal{V}}), S) = (F'_1, ..., F'_k)\).

1. By Definition 5, for all \(j\) such that \(1 \leq j \leq k\), \(\langle A_j \rangle^{\mathcal{V}} \subseteq F_j\). Now, suppose that \(\phi \in A_i\) for some \(A_i \in A\). Therefore, \(\langle \phi \rangle^{\mathcal{V}} \in F_i\). Hence, \(T \models_{A_i} \phi\) according to Definition 11.

2. Suppose that \(T \models_{A_i} \phi\), \(A_j \subseteq A'_j\) for all \(j\) such that \(1 \leq j \leq k\), and \(R' \subseteq R\). Then, by Definition 11, \(\langle \phi \rangle^{\mathcal{V}} \in F_i\). Since \(S = S'\), and \(R \subseteq R'\), then by Definition 10, \(\preceq_{\mathcal{TV}} \subseteq \preceq_{\mathcal{TV}'}\). Accordingly, it must be that, for all \(j\) such that \(1 \leq j \leq k\), \(F_j \subseteq F'_j\). Hence, \(\langle \phi \rangle^{\mathcal{V}} \in F'_i\) as well. By Definition 11, it must be that \(T' \models_{A_i} \phi\).

3. Suppose that \(A'_i = A_i \cup \{\psi\}\) for some \(i\) such that \(1 \leq i \leq k\), \(A'_j = A_j\) for all \(j \neq i\), and \(R' = R\). Further, suppose that \(T \models_{A_i} \psi\) and \(T' \models_{A_i} \phi\). Then, according to Definition 11, \(\langle \psi \rangle^{\mathcal{V}} \in F_i\) and \(\langle \phi \rangle^{\mathcal{V}} \in F'_i\). Since \(S = S'\) and \(R = R'\), then by Definition 10, \(\preceq_{\mathcal{TV}} \subseteq \preceq_{\mathcal{TV}'}\). Hence, according to Definition 5, it must be that \(F_l = F'_l\) for all \(l\) such that \(1 \leq l \leq k\). Hence, \(\langle \phi \rangle^{\mathcal{V}} \in F_i\). Then, according to Definition 11, \(T \models_{A_i} \phi\).

4. Suppose that \(A'_i = A_i \cup \{\phi\}\) for some \(i\) such that \(1 \leq i \leq k\), \(i \in S\), \(R' = R\), and \(T' \models_{A_i} \psi\). Then, by Definition 11, \(\langle \psi \rangle^{\mathcal{V}} \in F'_i\). Suppose now that \(T \models_{A_i} \neg (\phi \Rightarrow \psi)\). This means that \(T \models_{A_i} \phi \land \neg \psi\). According to Definition 11, \(\langle \phi \land \neg \psi \rangle^{\mathcal{V}} \in F_i\). By the definition of the interpretation function, this means that...
$[\phi]^V \cdot [\neg \psi]^V \in \mathcal{F}_i$. Since $i \in S$, then according to condition 2 of Definition 4, $[\phi]^V \in \mathcal{F}_i$ and $[\neg \psi]^V \in \mathcal{F}_i$. Since $S = S'$, and $R \subseteq R'$, then by Definition 10, $\preceq_T^V \subseteq \preceq_T^V$. It follows then that $\mathcal{F}_i \subseteq \mathcal{F}_i'$. But then we get a contradiction as both $[\psi]^V \in \mathcal{F}_i'$ and $[\neg \psi]^V \in \mathcal{F}_i'$. Hence, it must be that $T \models_{A_i} \phi \Rightarrow \psi$. 