Reasoning with Artificial Mental States: An Algebraic Approach

Appendix

Nourhan Ehab[1] and Haythem O. Ismail[2,1]

[1] German University in Cairo, Egypt
Department of Computer Science and Engineering
[2] Cairo University, Egypt
Department of Engineering Mathematics
{nourhan.ehab,haythem.ismail}@guc.edu.eg

In this document, we provide the proofs for the theorems in the paper *Reasoning with Artificial Mental States: An Algebraic Approach* that has been accepted for presentation at the 6^{th} Workshop on Formal and Cognitive Reasoning (FCR-2020).

Proof of Theorem 1

Let $\preceq$ be a k partial order on $\mathfrak{A}$ which is classical in i for some $i \in S$ and $\mathfrak{F}_{\preceq}(\langle \mathcal{Q}_1, ..., \mathcal{Q}_k \rangle, S) = \langle \mathcal{Q}'_1, ..., \mathcal{Q}'_k \rangle$. Then, by Definition 3, there does not exist an order $(a_1, ..., a_k) \leq (b_1, ..., b_k)$ such that $a_i \not\leq b_i$ for all i such that $1 \leq i \leq k$. Hence, $F(\mathcal{Q}_i) \subseteq \mathcal{Q}'_i$. By, Definition 3, there does not exist an order such that $(\{\top\}^{i-1} \times \{a\} \times \{\top\}^{k-i}) \times (\mathcal{P}^{i-1} \times \{b\} \times \mathcal{P}^{k-i}) \subseteq \preceq_k$ and $a_i \not\leq b_i$. Hence, $\mathcal{Q}'_i \subseteq F(\mathcal{Q}_i)$. Therefore, $\mathcal{Q}'_i \subseteq F(\mathcal{Q}_i)$.

Proof of Theorem 2

In the following, for every $\mathcal{L}$ valuation $\mathcal{V}$, let $\preceq_{\mathbb{T}^{\mathcal{V}}}$ be a ${\mathbb{T}'}^{\mathcal{V}}$-induced order and $\preceq_{\mathbb{T}'^{\mathcal{V}}}$ be a ${\mathbb{T}'}^{\mathcal{V}}$-induced order. Further, let the multifilter $\mathfrak{F}_{\preceq_{\mathbb{T}^{\mathcal{V}}}}(\langle [\![\mathbb{A}_1]\!]^{\mathcal{V}}, ..., [\![\mathbb{A}_k]\!]^{\mathcal{V}} \rangle, \mathbb{S}) = \langle \mathcal{F}_1, ...\mathcal{F}_k \rangle$ and $\mathfrak{F}_{\preceq_{\mathbb{T}'^{\mathcal{V}}}}(\langle [\![\mathbb{A}'_1]\!]^{\mathcal{V}}, ..., [\![\mathbb{A}'_k]\!]^{\mathcal{V}} \rangle, \mathbb{S}) = \langle \mathcal{F}'_1, ...\mathcal{F}'_k \rangle$.

1. By Definition 5, for all j such that $1 \leq j \leq k$, $[\![\mathbb{A}_j]\!]^{\mathcal{V}} \subseteq \mathcal{F}_j$. Now, suppose that $\phi \in \mathbb{A}_i$ for some $\mathbb{A}_i \in \mathbb{A}$. Therefore, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. Hence, $\mathbb{T} \models_{A_i} \phi$ according to Definition 11.
2. Suppose that $\mathbb{T} \models_{A_i} \phi$, $\mathbb{A}_j \subseteq \mathbb{A}'_j$ for all j such that $1 \leq j \leq k$, and $\mathbb{R}' \subseteq \mathbb{R}$. Then, by Definition 11, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. Since $\mathbb{S} = \mathbb{S}'$, and $\mathbb{R} \subseteq \mathbb{R}'$, then by Definition 10, $\preceq_{\mathbb{T}^{\mathcal{V}}} \subseteq \preceq_{\mathbb{T}'^{\mathcal{V}}}$. Accordingly, it must be that, for all j such that $1 \leq j \leq k$, $\mathcal{F}_j \subseteq \mathcal{F}'_j$. Hence, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F'}_i$ as well. By Definition 11, it must be that $\mathbb{T}' \models_{A_i} \phi$.
3. Suppose that $\mathbb{A}'_i = \mathbb{A}_i \cup \{\psi\}$ for some i such that $1 \leq i \leq k$, $\mathbb{A}'_j = \mathbb{A}_j$ for all $j \neq i$, and $\mathbb{R}' = \mathbb{R}$. Further, suppose that $\mathbb{T} \models_{A_i} \psi$ and $\mathbb{T}' \models_{A_i} \phi$. Then, according to Definition 11, $[\![\psi]\!]^{\mathcal{V}} \in \mathcal{F}_i$ and $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}'_i$. Since $\mathbb{S} = \mathbb{S}'$ and $\mathbb{R} = \mathbb{R}'$, then by Definition 10, $\preceq_{\mathbb{T}^{\mathcal{V}}} = \preceq_{\mathbb{T}'^{\mathcal{V}}}$. Hence, according to Definition 5, it must be that $\mathcal{F}_l = \mathcal{F}'_l$ for all l such that $1 \leq l \leq k$. Hence, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. Then, according to Definition 11, $\mathbb{T} \models_{A_i} \phi$.
4. Suppose that $\mathbb{A}'_i = \mathbb{A}_i \cup \{\phi\}$ for some i such that $1 \leq i \leq k$, $i \in \mathbb{S}$, $\mathbb{R}' = \mathbb{R}$, and $\mathbb{T}' \models_{A_i} \psi$. Then, by Definition 11, $[\![\psi]\!]^{\mathcal{V}} \in \mathcal{F}'_i$ $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}'_i$. Suppose now that $\mathbb{T} \models_{A_i} \neg(\phi \Rightarrow \psi)$. This means that $\mathbb{T} \models_{A_i} \phi \wedge \neg\psi$. According to Definition 11, $[\![\phi \wedge \neg\psi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. By the definition of the interpretation function, this means that

$[\![\phi]\!]^{\mathcal{V}} \cdot [\![\neg\psi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. Since $i \in \mathbb{S}$, then according to condition 2 of Definition 4, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}_i$ and $[\![\neg\psi]\!]^{\mathcal{V}} \in \mathcal{F}_i$. Since $\mathbb{S} = \mathbb{S}'$, and $\mathbb{R} \subseteq \mathbb{R}'$, then by Definition 10, $\preceq_{\mathbb{T}^{\mathcal{V}}} \subseteq \preceq_{\mathbb{T}'^{\mathcal{V}}}$. It follows then that $\mathcal{F}_i \subseteq \mathcal{F}'_i$. But then we get a contradiction as both $[\![\psi]\!]^{\mathcal{V}} \in \mathcal{F}'_i$ and $[\![\neg\psi]\!]^{\mathcal{V}} \in \mathcal{F}'_i$. Hence, it must be that $\mathbb{T} \models_{A_i} \phi \Rightarrow \psi$.